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ELEVEN LARGE-AMPLITUDE LIMIT CYCLES IN A POLYNOMIAL SYSTEM

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Abstract

In this paper, an indirect method is used to investigate the bifurcations of limit cycles at infinity for a class of seventh-degree polynomial system, in which the problem for bifurcations of limit cycles at infinity is transferred into that at the origin. By the computation of singular point values, the conditions of the origin (correspondingly infinity) to be a center and the highest degree fine focus are derived. Finally, it is showed firstly that a seventh-degree differential system can bifurcate eleven limit cycles at infinity.

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1. Introduction

The second part of Hilbert's 16th problem is concerned with the number and relative distributions of limit cycles of the polynomial system

$$\frac{dx}{dt} = P_n(x, y), \frac{dy}{dt} = Q_n(x, y), \tag{1}$$

where P_n and Q_n are Polynomials of degree n. Let H_n be the Hilbert number, then in [10, 11], it is known that $H_2 \geq 4, H_3 \geq 12$. To study distribution of limit cycles in the planar, we need to consider not only the case of finite critical points but also the case of infinity. In the second case, the research is mainly concerned on the following system of degree $2n + 1$

$$\frac{dx}{dt} = \sum_{k=0}^{2n} X_k(x, y) + (-y + \delta x)(x^2 + y^2)^n,$$

$$\frac{dy}{dt} = \sum_{k=0}^{2n} Y_k(x, y) + (\delta y + x)(x^2 + y^2)^n, \tag{2}$$

where $X_k(x, y), Y_k(x, y)$ are homogeneous polynomials of degree k in x, y. As indicated in [6] the equator Γ_∞ on the Poincaré closed sphere is a trajectory of this system, having no real singular point. Γ_∞ is also called infinity. Conveniently, we denote I_n the maximum number of possible limit cycles in a neighborhood of infinity (large-amplitude limit cycles) of n degree polynomial differential system in the form of (2). As far as the number of limit cycles bifurcated from infinity is concerned, there are some results so far in the literature as follows: in [1], Blows and Rousseau had studied bifurcations of limit cycle at infinity for a class of cubic system and obtained 5 limit cycles at infinity. In [8, 12], the authors gave a real planar cubic system, which bifurcated 7 limit cycles at infinity (i.e., $I_3 \geq 7$). For quintic systems, [2, 3, 13] show that $I_5 \geq 5$, $I_5 \geq 8$, $I_5 \geq 11$, respectively. For the higher degree systems, there are few results. $I_7 \geq 9$ is proved in [4]. In this paper, we consider a class of special seventh-degree differential system with the form

$$\frac{dx}{dt} = A_{10}x + B_{10}y + (B_{11} + A_{02})x^2 + A_{11}xy + A_{02}y^2$$
$$+ B_{30}x^3 - B_{21}x^2y + B_{12}xy^2$$
$$+ A_{03}y^3 + (A_{32}x - A_{23}y)(x^2 + y^2)^2 + (-y + \delta x)(x^2 + y^2)^3,$$
$$\frac{dy}{dt} = B_{10}y + B_{01}x + (A_{11} + B_{02})y^2 + B_{11}xy + B_{02}x^2$$
$$+ B_{30}y^3 + B_{21}y^2x + B_{12}yx^2$$
$$- A_{03}x^3 + (A_{23}x + A_{32}y)(x^2 + y^2)^2 + (x + \delta y)(x^2 + y^2)^3, \tag{3}$$

where $A_{ij}, B_{ij}, \delta \in R, i, j = 0, 1, 2$ and prove that $I_7 \geq 11$.

2. Some Preliminary Results

Consider a real differential system

$$\frac{d\xi}{dt} = -\delta\xi - \eta + \sum_{k=2}^{\infty} X_k(\xi, \eta), \frac{d\eta}{dt} = \xi - \delta\eta + \sum_{k=2}^{\infty} Y_k(\xi, \eta), \tag{4}$$

where $X_k(\xi, \eta), Y_k(\xi, \eta)$ are homogeneous polynomials of degree k in ξ, η. Under the polar coordinates $\xi = r\cos\theta, \eta = r\sin\theta$, system (4) can take the following form

$$\frac{dr}{d\theta} = r\frac{-\delta + \sum_{k=2}^{\infty} r^{k-1}\varphi_{k+1}(\theta)}{1 + \sum_{k=2}^{\infty} r^{k-1}\psi_{k+1}(\theta)}, \tag{5}$$

where

$$\varphi_{k+1}(\theta) = \cos\theta X_k(\cos\theta, \sin\theta) + \sin\theta Y_k(\cos\theta, \sin\theta),$$
$$\psi_{k+1}(\theta) = \cos\theta Y_k(\cos\theta, \sin\theta) - \sin\theta X_k(\cos\theta, \sin\theta),$$

$k = 1, 2, \ldots$.

For sufficient small h, let

$$d(h) = r(2\pi, h) - h, \; r = r(\theta, h) = \sum_{m=1}^{\infty} \nu_m(\theta) h^m \tag{6}$$

be the Poincaré succession function and the solution of Eq.(5) associated with the initial condition $r|_{\theta=0} = h$. It is evident that

$$\nu_1(\theta) = e^{\delta\theta}, \; \nu_m(0) = 0, \; m = 2, 3, \cdots. \tag{7}$$

Similar to [6], if $\nu_1(2\pi) \neq 1$, then the origin is called a rough focus; if $\nu_1(2\pi) = 1$, and $\nu_2(2\pi) = \nu_3(2\pi) = \cdots = \nu_{2k}(2\pi) = 0, \nu_{2k+1}(2\pi) \neq 0$, then the origin is called a weak focus (fine focus) of order k and the quantity of $\nu_{2k+1}(2\pi)$ is called the k-th focal value at the origin $(k = 1, 2, ...)$; if $\nu_1(2\pi) = 1$, and for any positive integer k, $\nu_{2k+1}(2\pi) = 0$, then the origin is called a center.

By means of transformation

$$z = \xi + i\eta, \; w = \xi - i\eta, \; T = it, \; i = \sqrt{-1}, \tag{8}$$

system $(4)|_{\delta=0}$ can be transformed into the following complex system

$$\frac{dz}{dT} = z + \sum_{k=2}^{\infty} Z_k(z, w) = Z(z, w), \; \frac{dw}{dT} = -w - \sum_{k=2}^{\infty} W_k(z, w) = -W(z, w), \tag{9}$$

where z, w, T are complex variables and

$$Z_k(z, w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^{\alpha} w^{\beta}, \; W_k(z, w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta}.$$

It is obvious that the coefficients of system (9) satisfy the conjugated condition, namely,

$$\overline{a}_{\alpha\beta} = b_{\alpha\beta}, \; \alpha \geq 0, \; \beta \geq 0, \; \alpha + \beta \geq 2. \tag{10}$$

It is called that system $(4)|_{\delta=0}$ and (9) are concomitant.

Lemma 2.1 (see [7]). *For system* (9), *we can derive successively the terms of the following formal series* $M(z, w) = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta}$ *such that*

$$\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = \sum_{m=1}^{\infty}(m+1)\mu(m)(zw)^k, \tag{11}$$

where $c_{0,0} = 1$, $c_{k,k}$ *is any real number,* $k = 1, 2, \cdots,$ *and for any integer* m, $\mu(m)$ *is determined by following recursive formulas*:

$$c_{0,0} = 1, \text{ if } (\alpha = \beta > 0) \text{ or } \alpha < 0 \text{ or } \beta < 0, \text{ then } c_{\alpha,\beta} = 0$$

else

$$c_{\alpha,\beta} = \frac{1}{\beta - \alpha}\sum_{k+j=3}^{\alpha+\beta+2}[(\alpha+1)a_{k,j-1} - (\beta+1)b_{j,k-1}]c_{\alpha-k+1,\beta-j+1},$$

$$\mu(m) = \sum_{k+j=3}^{2m+2}[a_{k,j-1} - b_{j,k-1}]c_{m-k+1,m-j+1}. \tag{12}$$

As in [7, 9], μ_k in Lemma 2.1 is called the *singular point value* at the origin of system (9). The relations between singular point values and focal values is given in the following lemma.

Lemma 2.2 ([8])**.** *For system* (9), *the first non-vanishing singular point value and the first non-vanishing focal value of its concomitant system* $(4)|_{\delta=0}$ *are related by*

$$v_{2m+1}(2\pi) = i\pi\mu_m. \tag{13}$$

3. Singular Point Values and Center Conditions

By means of translations $x = (\zeta + \varsigma)/2$, $y = -i(\zeta - \varsigma)/2$, $t = i\tau$, $i = \sqrt{-1}$, we can obtain a concomitant complex seventh-degree system to system (3)

$$\frac{d\zeta}{d\tau} = a_{10}\zeta + a_{01}\varsigma + a_{20}\zeta^2 + a_{11}\zeta\varsigma + a_{21}\zeta^2\varsigma + a_{03}\varsigma^3$$
$$+ a_{32}\zeta^3\varsigma^2 + (1 - i\delta)\zeta^4\varsigma^3,$$

$$\frac{d\varsigma}{d\tau} = -[b_{10}\varsigma + b_{01}\zeta + b_{20}\varsigma^2 + b_{11}\zeta\varsigma + b_{21}\varsigma^2\zeta + b_{03}\zeta^3 + b_{32}\varsigma^3\zeta^2 + (1 + i\delta)\zeta^4\varsigma^3], \tag{14}$$

where

$$a_{10} = \frac{-A_{01} - iA_{10} + B_{01} - iB_{10}}{2}, \quad b_{10} = \frac{-A_{01} + iA_{10} + B_{01} + iB_{10}}{2},$$

$$a_{01} = \frac{A_{01} - iA_{10} + B_{01} + iB_{10}}{2}, \quad b_{01} = \frac{A_{01} + iA_{10} + B_{01} - iB_{10}}{2},$$

$$a_{20} = \frac{-A_{11} - iB_{11}}{2}, \quad b_{20} = \frac{-A_{11} + iB_{11}}{2},$$

$$a_{11} = \frac{-2iA_{02} - iB_{11} + 2B_{20} + A_{11}}{2}, \quad b_{11} = \frac{2iA_{02} + iB_{11} + 2B_{20} + A_{11}}{2},$$

$$a_{03} = \frac{-A_{03} + B_{21} - iB_{30} + iB_{12}}{4}, \quad b_{03} = \frac{-A_{03} + B_{21} + iB_{30} - iB_{12}}{4},$$

$$a_{21} = \frac{-3A_{03} - iA_{12} - 3iB_{30} + B_{21}}{4}, \quad b_{21} = \frac{-3A_{03} + iA_{12} + 3iB_{30} + iB_{21}}{4},$$

$$a_{32} = A_{23} - iA_{32}, \quad b_{32} = A_{23} + iA_{32}. \tag{15}$$

Evidently, the coefficients of system (14) satisfy the conjugate condition, i.e., $a_{ij} = \overline{b_{ij}}(i, j = 0, 1, 2, 3)$. By means of coordinate $\zeta = \frac{z}{(zw)^4}$, $\varsigma = \frac{w}{(zw)^4}$, and time scaling $d\tau = \frac{1}{(\zeta\varsigma)^3}dT$, system (14) can be transformed into the following system

$$\frac{dz}{dT} = \left(1 + \frac{1}{7}i\delta\right)z + \left(\frac{3}{7}a_{32} + \frac{4}{7}b_{32}\right)z^8w^7 + \frac{3}{7}a_{03}z^{13}w^{16} + \left(\frac{3}{7}a_{21} + \frac{4}{7}b_{21}\right)z^{15}w^{14} + \frac{4}{7}b_{03}z^{17}w^{12} + \left(\frac{3}{7}a_{11} + \frac{4}{7}b_{20}\right)z^{18}w^{18}$$
$$+\left(\frac{3}{7}a_{20} + \frac{4}{7}b_{11}\right)z^{19}w^{17} + \frac{3}{7}a_{01}z^{21}w^{22} + \left(\frac{3}{7}a_{10} + \frac{4}{7}b_{10}\right)z^{22}w^{21} + \frac{4}{7}b_{01}z^{23}w^{20},$$

$$\frac{dw}{dT} = -\Big[\Big(1 - \frac{1}{7}i\delta\Big)w + \Big(\frac{4}{7}a_{32} + \frac{3}{7}b_{32}\Big)w^8z^7 + \frac{3}{7}b_{03}w^{13}z^{16}$$

$$+ \Big(\frac{4}{7}a_{21} + \frac{3}{7}b_{21}\Big)w^{15}z^{14} + \frac{4}{7}a_{03}w^{17}z^{12} + \Big(\frac{4}{7}a_{20} + \frac{3}{7}b_{11}\Big)w^{18}z^{18}$$

$$+ \Big(\frac{4}{7}a_{11} + \frac{3}{7}b_{20}\Big)w^{19}z^{17} + \frac{3}{7}b_{01}w^{21}z^{22}$$

$$+ \Big(\frac{4}{7}a_{10} + \frac{3}{7}b_{10}\Big)w^{22}z^{21} + \frac{4}{7}a_{01}w^{23}z^{20}\Big]. \quad (16)$$

Suppose that system (16’) be concomitant system of system (16) (i.e, by means of transformation $z = \xi + \eta i,\ w = \xi - \eta i,\ T = it,\ i = \sqrt{-1}$, system (16) can become system (16’)), then the problem of bifurcation at infinity in real system (3) will be transformed into that at the origin of real system (16’).

According to recursive formulas given by Lemma 2.1, and using computer algebra system–Mathematica, we compute the singular point values of the origin of system $(16)|_{\delta=0}$ and simplify them, we have

Theorem 3.1. *For system* $(16)|_{\delta=0}$, *the first* 112 *singular point values of the origin as follows*

$$\mu(7) = \frac{-a_{32} + b_{32}}{7},\ \mu(14) = \frac{-a_{21} + b_{21}}{7},\ \mu(21) = \frac{-a_{10} + b_{10}}{7},\ \mu(28) = 0,$$

$$\mu(35) = \frac{a_{11}a_{20} - b_{11}b_{20}}{7};$$

Case1. $a_{20}b_{20} = 0$,

$$\mu(42) = \mu(49) = 0,\ \mu(56) = \frac{a_{11}^2b_{01} - b_{11}^2a_{01}}{7},\ \mu(63) = 0,$$

$$\mu(70) = \frac{-a_{01}a_{11}^2b_{03} + b_{01}b_{11}^2a_{03}}{21},$$

$$\mu(77) = \mu(84) = 0,\ \mu(91) = \frac{(-a_{11}^4b_{03} + b_{11}^4a_{03})(a_{32} + b_{32})}{12},$$

$$\mu(98) = \frac{5(-a_{11}^4 b_{03} + b_{11}^4 a_{03})(a_{21} + b_{21})}{21},$$

$$\mu(105) = \frac{27(-a_{11}^4 b_{03} + b_{11}^4 a_{03})(a_{10} + b_{10})}{56},$$

$$\mu(112) = \frac{5a_{03}b_{03}(-a_{11}^4 b_{03} + b_{11}^4 a_{03})}{63};$$

Case 2. $a_{20}b_{20} \neq 0$,

$$\mu(42) = \mu(49) = 0,$$

$$\mu(56) = \frac{-4(a_{20}^2 a_{01} - b_{20}^2 b_{01}) + (a_{11}^2 b_{01} - b_{11}^2 a_{01}) + 4(a_{20}a_{01}b_{11} - b_{20}b_{01}a_{11})}{7};$$

Case 2.1. $a_{11} = 2b_{20}$, $b_{11} = 2a_{20}$,

$$\mu(63) = \mu(70) = \mu(77) = \mu(84) = \mu(91) = \mu(98) = \mu(105) = \mu(122) = 0;$$

Case 2.2. $a_{11} \neq 2b_{20}$, $b_{11} \neq 2a_{20}$,

$$\mu(63) = 0,\ \mu(70) = [2(a_{03}a_{20}^2 b_{01} - b_{03}b_{20}^2 a_{01}) - 5(a_{03}a_{20}^2 b_{01}b_{11} - b_{03}b_{20}^2 a_{01}a_{11}) - 2(b_{03}a_{11}^2 a_{01} - a_{03}b_{11}^2 b_{01})] / 42;$$

Case 2.2.1. if $a_{03}a_{20}^4 = b_{03}b_{20}^4$, then

$$\mu(77) = \mu(84) = \mu(91) = \mu(98) = \mu(105) = \mu(112) = 0;$$

Case 2.2.2. If $a_{03}a_{20}^4 \neq b_{03}b_{20}^4$, $a_{11} = \frac{1}{2}b_{20}$, $b_{11} = \frac{1}{2}a_{20}$, then

$$\mu(77) = \frac{33(a_{03}a_{20}^2 b_{01} - b_{03}b_{20}^2 a_{01})(a_{32} + b_{32})}{448},$$

$$\mu(84) = 9[5(b_{03}b_{20}^4 - a_{03}a_{20}^4) - (a_{01}a_{21}b_{03}b_{20}^2 - b_{01}b_{21}a_{03}a_{20}^2) + (a_{03}b_{01}a_{20}^2 a_{21} - b_{03}a_{01}b_{20}^2 b_{21})] / 56,$$

$$\mu(91) = \frac{3}{2240}[195(a_{03}a_{10}a_{20}^2 b_{01} - b_{03}b_{10}b_{20}^2 a_{01})$$

$$+ 28(a_{03}a_{20}b_{01}^2 b_{20} - b_{03}b_{20}a_{01}^2 a_{20})$$

$$- 195(a_{01}a_{10}b_{03}b_{20}^2 - b_{01}b_{10}a_{03}a_{20}^2)],$$

$$\mu(98) = 9[175(a_{01}a_{03}a_{20}^6 - b_{01}b_{03}b_{20}^6) + (a_{01}a_{03}^2 a_{20}^2 b_{01}^2 b_{03}$$

$$- b_{01}b_{03}^2 b_{20}^2 a_{01}^2 a_{03})] \Big/ 784 a_{01}b_{01};$$

Case 2.2.3. If $a_{03}a_{20}^4 \neq b_{03}b_{20}^4$, $a_{11} \neq \frac{1}{2}b_{20}$, $b_{11} \neq \frac{1}{2}a_{20}$, then

$\mu(77) = 0,$

$$\mu(84) = \frac{1}{7}[-16(a_{03}a_{20}^4 - b_{03}b_{20}^4) + 28(a_{03}a_{20}^3 b_{11} - b_{03}b_{20}^3 a_{11})$$

$$+ 3(a_{03}a_{20}b_{11}^3 - b_{03}b_{20}a_{11}^3)$$

$$- 16(a_{03}a_{20}^2 b_{11}^2 - b_{03}b_{20}^2 a_{11}^2)],$$

$$\mu(91) = \frac{187(-a_{03}a_{20}^4 + b_{03}b_{20}^4)(a_{32} + b_{32})}{13608},$$

$$\mu(98) = \frac{40(-a_{03}a_{20}^4 + b_{03}b_{20}^4)(a_{21} + b_{21})}{1701},$$

$$\mu(105) = \frac{5(-a_{03}a_{20}^4 + b_{03}b_{20})(a_{10} + b_{10})}{168},$$

$$\mu(112) = \frac{17605 a_{03}b_{03}(a_{03}a_{20}^4 - b_{03}b_{20}^4)}{35721},$$

where $\mu(k) = 0$, $k \neq 7i$, $i \leq 16$, $i \in N$. *In the above expression of* $\mu(k)$, *we have applied the conditions* $\mu(1) = \mu(2) = \cdots = \mu(k-1) = 0$, *for* $k = 2, 3, \cdots, 112$.

Theorem 3.2. *In system* $(16)|_{\delta=0}$, *all of the first* 112 *singular point values at the origin vanish, if and only if one of the following conditions holds*

(1) $a_{32} = b_{32},\ a_{21} = b_{21},\ a_{10} = b_{10},\ a_{11}a_{20} = b_{11}b_{20},\ a_{11}^4 b_{03} = b_{11}^4 a_{03},$

$$a_{20}b_{20} = 0; \tag{17}$$

(2) $a_{21} = b_{21}, a_{10} = b_{10}, a_{11}a_{20} = b_{11}b_{20}, a_{01}a_{20}^2 = b_{01}b_{20}^2, a_{03}a_{20}^4$

$$= b_{03}b_{20}^4,\ a_{32} = b_{32},\ a_{20}b_{20} \neq 0; \tag{18}$$

(3) $a_{32} = b_{32}, a_{21} = b_{21}, a_{10} = b_{10}, a_{11} = 2b_{20},$

$$b_{11} = 2a_{20}, a_{20}b_{20} \neq 0. \tag{19}$$

Proof. Obviously, the condition is sufficient. Now we prove its necessity.

From $\mu(35) = (a_{20}a_{11} - b_{20}b_{11})/7 = 0,$ we can know that $a_{20}b_{20} = 0$ or $a_{20}b_{20} \neq 0.$

Case 1. $a_{20}b_{20} = 0.$ By $\mu(56) = (b_{01}a_{11}^2 = a_{01}b_{11}^2)/7 = 0,$ it can be obtained that $a_{11}b_{11} = 0$ or $a_{11}b_{11} \neq 0.$ (i) If $a_{11}b_{11} = 0,$ then $\mu(63) = \mu(70) = \cdots = \mu(112) = 0;$ (ii) if $a_{11}b_{11} \neq 0,$ there exists constant p, such that $a_{01} = pa_{11}^2, b_{01} = pb_{11}^2,$ so $\mu(63) = 0, \mu(70) = (-a_{11}^4 b_{03} + b_{11}^4 a_{03})$ $p/21.$ If $a_{11}^4 b_{03} = b_{11}^4 a_{03},$ then $\mu(77) = \mu(84) = \cdots = \mu(112) = 0,$ else $a_{11}^4 b_{03} \neq b_{11}^4 a_{03},$ i.e., $a_{03}b_{03} \neq 0,$ then $\mu(112) = -5a_{03}b_{03}(-a_{11}^4 b_{03} + b_{11}^4 b_{03})/63 \neq 0.$ Therefore, the condition (1) holds.

Case 2. $a_{20}b_{20} \neq 0.$ If there exists constant r, such that $a_{11} = rb_{20},\ b_{11} = ra_{20},$ then $\mu(56) = (-a_{01}a_{20}^2 + b_{01}b_{20}^2)(-2 + r)^2,$ so we can obtain $r = 2$ or $a_{01}a_{20}^2 = b_{01}b_{20}^2.$ (I) if $r = 2,$ then $\mu(63) = \mu = (70) = \cdots = \mu(112) = 0,$ so condition (3) holds; (II) if $r \neq 2,$ there exists constant q, such that $a_{01} = qb_{20}^2, b_{01} = qa_{20}^2.$ From $\mu(70) = (a_{03}a_{20}^4 - b_{03}b_{20}^4)q(-2 + r)(-1 + 2r),$ we can know that $a_{03}a_{20}^4 = b_{03}b_{20}^4$ or $q = 0$ or $r = 2$ or $r = 1/2.$ By computation and deduction, the condition (2) holds.

From Theorem 3.2, it can be got that

Theorem 3.3. *For system* $(16)|_{\delta=0}$, *all of singular point values at the origin are zero if and only if one of the three conditions of Theorem* 3.2 *holds. Hence the conditions of Theorem* 3.2 *are the center conditions of system* $(16)|_{\delta=0}$ *at the origin. Relevantly, the three conditions of Theorem* 3.2 *are the center conditions of system* $(14)|_{\delta=0}$ *at infinity*.

Proof. Let us prove the sufficiency. If the condition (17) or (18) holds, According to the Constructive Theorem of Singular point values(see ([12, Theorem 2.5] or [13, Theorem 4.15])), we get that all $\mu_k = 0$, $k = 1, 2, \cdots$. So, the origin of $(16)|_{\delta=0}$ is a complex center.

If condition (19) holds, system $(16)|_{\delta=0}$ has a analytic first integral

$$F(z, w) = w^7 z^7 G^{-\frac{1}{4}},$$

$$\begin{aligned} G &= 3 + 4b_{32}w^7z^7 + 12b_{10}w^{21}z^{21} + 6b_{21}w^{14}z^{14} \\ &+ 6b_{01}w^{20}z^{22} + 6a_{01}w^{22}z^{20} + 3a_{03}w^{16}z^{12} \\ &+ 12a_{20}w^{17}z^{18} + 3b_{03}w^{12}z^{16} + 12b_{20}w^{18}z^{17}, \end{aligned}$$

so the origin of $(16)|_{\delta=0}$ is also a center.

From Theorem 3.3, we have the following

Theorem 3.4. *Infinity of system* (3) *is a center if and only if* $\delta = 0$ *and one of the three conditions in Theorem* 3.3 *holds*.

4. Bifurcations of Limit Cycles

In order to consider bifurcations of limit cycles, system (16') needs to be transformed by transformation: $\xi = r\cos\theta$, $\eta = r\sin\theta$. And under the initial-value condition $r|_{\theta=0} = h$, its solution can be expressed by $r = \tilde{r}(\theta, h) = \sum_{m=1}^{\infty} \nu_{14m+1}(\theta)h^m$, where $\nu_1(\theta) = e^{\delta\theta}$, $v_m(0) = 0$, $m = 2, 2, \cdots$. The idea of finding m limit cycles around the origin of system (16') is as follows. First, we find conditions such that $v_1(2\pi) = 1$, $v_{15}(2\pi) = \cdots = v_{14k-1}(2\pi) = 0$, but $v_{14k+1}(2\pi) \neq 0$ for some $k \geq m$, then perform appropriate small perturbations to prove either that the Poincar'e succession function

$$d(h) = r(2\pi, h) - h = (v_1(2\pi) - 1)h + \sum_{j=1}^{\infty} v_{14j+1}(2\pi)h^{14j+1}, \tag{20}$$

has just m simple positive roots or that

$$v_{14j+1}(2\pi)v_{14(j-1)+1}(2\pi) < 0, \ |v_{14(j-1)+1}(2\pi)| \ll |v_{14j+1}(2\pi)|,$$
$$j = j_1, j_2, \cdots, j_m, \{j_1, j_2, \cdots, j_m\} \subset \{1, 2, \cdots, k\} \tag{21}$$

holds.

From Theorems 3.1 ~ 3.2 , we obtain

Theorem 4.1. *The highest order of singular point of system* $(16)|_{\delta=0}$ *at origin is* 112, *namely,* $\mu(1) = \mu(2) = \cdots = \mu(111) = 0, \mu(112) \neq 0,$ *if and only if one of the following conditions holds*

(1) $a_{21} = b_{21} = 0, \quad a_{10} = b_{10} = 0, \quad a_{32} = b_{32} = 0, \quad a_{01} = b_{01} = 0,$

$$a_{11} = \frac{4}{3}b_{20}, \quad b_{11} = \frac{4}{3}a_{20}, \quad a_{20}b_{20} \neq 0, \quad a_{03}a_{20}^4 \neq b_{03}b_{20}^4; \tag{22}$$

(2) $a_{21} = b_{21} = 0, \quad a_{10} = b_{10} = 0, \quad a_{32} = b_{32} = 0, \quad a_{01} = b_{01} = 0,$

$$a_{20} = b_{20} = 0, \quad a_{11}a_{20} = b_{11}b_{20}, \quad a_{03}b_{11}^4 \neq b_{03}a_{11}^4. \tag{23}$$

They are also the necessary and sufficient conditions for the origin of system $(16')|_{\delta=0}$ *to be a weak focus of order* 112.

By Theorem 3.1 and 4.1, the case of bifurcations of limit cycles is constructed as follows:

Theorem 4.2. *If the coefficients in system* (16) *satisfy*

$$\delta = -\varepsilon_{11}, a_{10} = -\varepsilon_1 + i\varepsilon_8, b_{10} = \overline{a}_{10}, a_{21} = \varepsilon_2 - i\varepsilon_9, b_{21} = \overline{a}_{21},$$

$$a_{11} = \frac{4}{3} - \varepsilon_4 + i\varepsilon_7, b_{11} = \overline{a}_{11}, a_{32} = -\varepsilon_3 + i\varepsilon_{10}, b_{32} = \overline{a}_{32} = -\varepsilon_5 + i\varepsilon_6,$$

$$b_{01} = \overline{a}_{01}, a_{20} = b_{20} = 1, a_{03} = i, b_{03} = -i,$$

where $\varepsilon_i (i = 1, 2, \cdots, 11)$ *is the small parameter which satisfy* $0 < \varepsilon_{11} \ll \varepsilon_{10} \ll \cdots \ll \varepsilon_2 \ll \varepsilon_1 \ll 1,$ *then system* (16') *has* 11 *limit cycles in a sufficiently small neighborhood of the origin, correspondingly, system* (3) *has* 11 *limit cycles in a sufficiently small neighborhood of infinity.*

Proof. According to Theorem 3.1, $\nu_1(2\pi) - 1 = e^{2\pi\delta} - 1$ and Lemma 2.2, by computation we have

$$\nu_1(2\pi) - 1 = -2\pi\varepsilon_{11} + o(\varepsilon_{11}),$$

$$\nu_{15}(2\pi) = \left[\frac{2}{7}\pi + \omega_1(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_{10} + o(\varepsilon_{10}),$$

$$\nu_{29}(2\pi) = \left[-\frac{2}{7}\pi + \omega_2(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_9 + o(\varepsilon_9),$$

$$\nu_{43}(2\pi) = \left[\frac{2}{7}\pi + \omega_3(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_8 + o(\varepsilon_8),$$

$$\nu_{71}(2\pi) = \left[-\frac{2}{7}\pi + \omega_4(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_7 + o(\varepsilon_7),$$

$$\nu_{113}(2\pi) = \left[\frac{8}{63}\pi + \omega_5(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_6 + o(\varepsilon_6),$$

$$\nu_{141}(2\pi) = \left[-\frac{10}{189}\pi + \omega_6(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_5 + o(\varepsilon_5),$$

$$\nu_{169}(2\pi) = \left[\frac{8}{21}\pi + \omega_7(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_4 + o(\varepsilon_4),$$

$$\nu_{183}(2\pi) = \left[-\frac{187}{3402}\pi + \omega_8(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_3 + o(\varepsilon_3),$$

$$\nu_{197}(2\pi) = \left[\frac{160}{1701}\pi + \omega_9(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_2 + o(\varepsilon_2),$$

$$\nu_{211}(2\pi) = \left[-\frac{5}{42}\pi + \omega_{10}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})\right]\varepsilon_1 + o(\varepsilon_1),$$

$$\nu_{255}(2\pi) = \frac{3520}{35721}\pi + o(1), \tag{24}$$

where $\omega_i(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10})$ is analytic at $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and $\omega_i(0, 0, 0, 0, 0, 0, 0, 0, 0, 0) = 0$, $i = 1, 2, \cdots, 10$.

From formula (24), we get that $\nu_{14(m-1)+1}(2\pi)\nu_{14m+1}(2\pi) < 0$ and $|\nu_{14(m-1)+1}(2\pi)| \ll |\nu_{14m+1}(2\pi)|$ $(m = 1, 2, 3, 5, 8, 10, 12, 13, 14, 15\ 16)$.

According to classical theory of Bautin, system (16') has 11 limit cycles in the sufficiently small neighborhood of the origin. Correspondingly, system (3) has 11 limit cycles in the sufficiently small neighborhood of infinity.

From Theorem 4.2, we get $I_7 \geq 11$.

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